

A new family of time-space harmonic polynomials with respect to Lévy processes

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Abstract

By means of a symbolic method, a new family of time-space harmonic polynomials with respect to Lévy processes is given. The coefficients of these polynomials involve a formal expression of Lévy processes by which many identities are stated. We show that this family includes classical families of polynomials such as Hermite polynomials. Poisson-Charlier polynomials result to be a linear combinations of these new polynomials, when they have the property to be time-space harmonic with respect to the compensated Poisson process. The more general class of Lévy-Sheffer polynomials is recovered as a linear combination of these new polynomials, when they are time-space harmonic with respect to Lévy processes of very general form. We show the role played by cumulants of Lévy processes so that connections with boolean and free cumulants are also stated.

keywords: time-space harmonic polynomial, Lévy process, cumulant, Lévy-Sheffer polynomial, umbral calculus

1 Introduction

A family of polynomials $\{P(x, t)\}_{t \geq 0}$ is said to be *time-space harmonic* with respect to a stochastic process $\{X_t\}_{t \geq 0}$ if $E[P(X_t, t) \mid \mathfrak{F}_s] = P(X_s, s)$, for all $s \leq t$, where $\mathfrak{F}_s = \sigma(X_\tau : \tau \leq s)$ is the natural filtration associated with $\{X_t\}_{t \geq 0}$. For random walks $\{X_n\}_{n \geq 0}$, Neveu [10] characterizes the family of time-space harmonic polynomials as the coefficients of the Taylor expansion

$$\frac{\exp\{zX_n\}}{E[\exp\{zX_n\}]} = \sum_{k \geq 0} R_k(X_n, n) \frac{z^k}{k!} \quad (1.1)$$

in some neighborhood of the origin. If $\{X_n\}_{n \geq 0}$ is replaced by a Lévy process $\{X_t\}_{t \geq 0}$, the left-hand side of (1.1) is the so-called Wald's exponential martingale [9]. The usefulness of time-space harmonic polynomials with respect to Lévy processes is that the stochastic process $\{P(X_t, t)\}$ is a martingale, whereas $\{X_t\}$ does not necessarily have this property.

The Wald's exponential martingale is well defined only when the process admits moment generating function $E[\exp\{zX_t\}]$ in a suitable neighborhood of the origin. Different authors have

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tried to overcome this gap by using other tools. Sengupta [15] uses a discretization procedure to extend the results proved by Goswami and Sengupta in [8]. Solé and Utzet [17] use Ito's formula showing that time-space harmonic polynomials with respect to Lévy processes are linked to exponential complete Bell polynomials [1]. The Wald's exponential martingale (1.1) has been recently reconsidered also in [16], but without this giving rise to a closed expression for these polynomials.

In this paper, by using a symbolic method, known in the literature as the *classical umbral calculus*, we give a new family of time-space harmonic polynomials, which could be easily implemented in any symbolic software, see [5] as example. Thanks to the results in [4], we show that this new family includes and generalizes the exponential complete Bell polynomials.

The classical umbral calculus we use consists essentially in a moment symbolic calculus, since its basic device is to represent an unital sequence of numbers by a symbol α , named *umbra*, i.e. to associate the sequence $1, a_1, a_2, \dots$ to the sequence $1, \alpha, \alpha^2, \dots$ of powers of α through an operator E that looks like the expectation of random variables [11]. The n -th element of the sequence is the n -th moment of α . As a matter of fact, an umbra looks the framework of a random variable with no reference to any probability space.

In this paper, we define an operator $E[\cdot|\alpha]$ that acts like the well-known conditional expectation of random variables. The umbral version of Lévy processes we propose takes into account their infinite divisible property. As corollaries, many identities are given on the coefficients of this family of time-space harmonic polynomials with respect to random walks and Lévy processes. Moreover, this expression allows us also to emphasize the role played by cumulants and to include boolean and free cumulants [3, 6]. As example, we show that this new family includes Hermite polynomials which are time-space harmonic with respect to Brownian motion. We prove that Poisson-Charlier polynomials are linear combinations of the introduced polynomials when they are time-space harmonic with respect to compensated Poisson processes. We show that the more general class of Lévy-Sheffer polynomials [13, 14] are linear combinations of the introduced polynomials and we characterize a general form for the associated Lévy processes.

The paper is structured as follows. Section 2 is provided for readers unaware of the classical umbral calculus. Let us underline that the theory of the classical umbral calculus has now reached a more advanced level compared to the elements here resumed. We have chosen to recall terminology, notation and the basic definitions strictly necessary to deal with the object of this paper. In Section 3 the new notion of conditional evaluation with respect to umbrae is introduced, which is the key to characterize time-space harmonic polynomials in terms of umbrae. Examples and applications are introduced in Section 4, where special emphasis is devoted to the role played by cumulants in the expression of these polynomials.

2 The classical umbral calculus

In the following, terminology, notation and some basic definitions of the classical umbral calculus are recalled. We skip any proof: the reader interested in deeper analysis is referred to the papers [2, 4].

Let $\mathbb{R}[x]$ be the ring of polynomials with real coefficients in the indeterminate x . The classical umbral calculus is a syntax with an alphabet $\mathcal{A} = \{\alpha, \beta, \gamma, \dots\}$ of elements, called *umbrae*, and a linear functional $E : \mathbb{R}[x][\mathcal{A}] \longrightarrow \mathbb{R}[x]$, called *evaluation*, such that $E[1] = 1$ and

$$E[x^n \alpha^i \beta^j \dots \gamma^k] = x^n E[\alpha^i] E[\beta^j] \dots E[\gamma^k] \quad (\text{uncorrelation property})$$

where $\alpha, \beta, \dots, \gamma$ are distinct umbrae and n, i, j, \dots, k are nonnegative integers.

A sequence $\{a_n\}_{n \geq 0} \in \mathbb{R}[x]$, with $a_0 = 1$, is *umbrally represented* by an umbra α if $E[\alpha^n] = a_n$, for all nonnegative integers n . Recall that a_n is called the *n-th moment* of α . An umbra is *scalar* if its moments are elements of \mathbb{R} while it is *polynomial* if its moments are polynomials of $\mathbb{R}[x]$. Special scalar umbrae are:

- i) the *augmentation* umbra ϵ , such that $E[\epsilon^n] = \delta_{0,n}$,¹ for all nonnegative integers n ;
- ii) the *unity* umbra u , such that $E[u^n] = 1$, for all nonnegative integers n ;
- iii) the *Bell* umbra β , whose moments are the Bell numbers;
- iv) the *singleton* umbra χ such that $E[\chi] = 1$ and $E[\chi^n] = 0$, for all integers $n > 1$.

The core of this moment symbolic calculus consists in the definition of the *dot-product* of two umbrae, which is fundamental both in the construction of time-space harmonic polynomials and in their applications. We recall in short the steps necessary to give this definition.

First let us remark that in the alphabet \mathcal{A} two (or more) distinct umbrae may represent the same sequence of moments. More formally, two umbrae α and γ are said to be *similar* when $E[\alpha^n] = E[\gamma^n]$ for all nonnegative integers n , in symbols $\alpha \equiv \gamma$. Therefore, given a sequence $\{a_n\}_{n \geq 0}$, there are infinitely many distinct, and thus similar umbrae, representing this sequence.

Now, define the symbol $n.\alpha$ representing $\alpha' + \alpha'' + \dots + \alpha'''$, where $\{\alpha', \alpha'', \dots, \alpha'''\}$ is a set of n uncorrelated umbrae similar to α . The symbol $n.\alpha$ is an example of *auxiliary umbra*. In a *saturated* umbral calculus, the auxiliary umbrae are treated as they were elements of \mathcal{A} [11]. The umbra $n.\alpha$ is called the *dot-product* of the integer n and the umbra α . Its moments are [2]:

$$q_i(n) = E[(n.\alpha)^i] = \sum_{k=1}^i (n)_k B_{i,k}(a_1, a_2, \dots, a_{i-k+1}), \quad (2.1)$$

where $(n)_k$ is the lower factorial and $B_{i,k}$ are the exponential partial Bell polynomials [1].

In (2.1), the polynomial $q_i(n)$ is of degree i in n . If the integer n is replaced by $t \in \mathbb{R}$, in (2.1) we have $q_i(t) = \sum_{k=1}^i (t)_k B_{i,k}(a_1, a_2, \dots, a_{i-k+1})$. We denote by $t.\alpha$ the auxiliary umbra such that $E[(t.\alpha)^i] = q_i(t)$, for all nonnegative integers i . The umbra $t.\alpha$ is the dot-product of t and α . Among its properties, we just recall the distributive property:

$$(t + s).\alpha \equiv t.\alpha + s.\alpha', \quad s, t \in \mathbb{R} \quad (2.2)$$

where $\alpha' \equiv \alpha$. In particular in (2.1) we can replace n with $-t$ by obtaining the auxiliary umbra $-t.\alpha$ with the remarkable property

$$-t.\alpha + t.\alpha' \equiv \epsilon, \quad (2.3)$$

where $\alpha' \equiv \alpha$. Due to property (2.3), the umbra $-t.\alpha$ is named the *inverse* umbra of $t.\alpha$.²

Let us consider again the polynomial $q_i(t)$ and suppose to replace t by an umbra γ . The polynomial $q_i(\gamma)$ is an *umbral polynomial* in $\mathbb{R}[x][\mathcal{A}]$, with support $\text{supp}(q_i(\gamma)) = \{\gamma\}$. Recall that the support $\text{supp}(p)$ of an umbral polynomial $p \in \mathbb{R}[x][\mathcal{A}]$ is the set of all umbrae occurring in it. The *dot-product* of γ and α is the auxiliary umbra $\gamma.\alpha$ such that $E[(\gamma.\alpha)^i] = E[q_i(\gamma)]$ for all

¹The symbol $\delta_{i,j}$ denotes the *Kronecker delta*, that is $\delta_{i,j} = 1$ if $i = j$, otherwise $\delta_{i,j} = 0$.

²Since $-t.\alpha$ and $t.\alpha$ are two distinct symbols, they can be considered uncorrelated, therefore $-t.\alpha + t.\alpha' \equiv -t.\alpha + t.\alpha \equiv \epsilon$. When no confusion occurs, we will use this last similarity instead of (2.3).

nonnegative integers i . Two umbral polynomials p and q are said to be *umbrally equivalent* if $E[p] = E[q]$, in symbols $p \simeq q$. Therefore equation (2.1), with n replaced by an umbra γ , can be written as $q_i(\gamma) \simeq (\gamma \cdot \alpha)^i \simeq \sum_{k=1}^i (\gamma)_k B_{i,k}(a_1, a_2, \dots, a_{i-k+1})$. Special dot-product umbrae are the α -cumulant umbra $\chi \cdot \alpha$ and α -partition umbra $\beta \cdot \alpha$, that we will use later on. Recall that if α, γ and η are uncorrelated umbrae, then

$$(\alpha + \eta) \cdot \gamma \equiv \alpha \cdot \gamma + \eta \cdot \gamma. \quad (2.4)$$

3 Time-space harmonic polynomials

Denote by \mathcal{X} the set $\mathcal{X} = \{\alpha\}$.

Definition 3.1. *The linear operator $E(\cdot | \alpha) : \mathbb{R}[x][\mathcal{A}] \rightarrow \mathbb{R}[\mathcal{X}]$ such that*

$$i) \ E(1 | \alpha) = 1;$$

$$ii) \ E(x^m \alpha^n \gamma^i \delta^j \dots | \alpha) = x^m \alpha^n E[\gamma^i] E[\delta^j] \dots \text{ for uncorrelated umbrae } \alpha, \gamma, \delta, \dots \text{ and for non-negative integers } m, n, i, j, \dots$$

is called conditional evaluation with respect to α .

In other words, Definition 3.1 says that the conditional evaluation with respect to α handles the umbra α as it was an indeterminate. The proofs of the following Propositions are straightforward taking into account Definition 3.1.

Proposition 3.2. *If $\alpha \in \mathcal{A}$ and $p \in \mathbb{R}[x][\mathcal{A}]$ with $\alpha \notin \text{supp}(p)$, then $E(p | \alpha) = E[p]$.*

Corollary 3.3. *If $\alpha \in \mathcal{A}$ and $p \in \mathbb{R}[x][\mathcal{A}]$, then $E[E(p | \alpha)] = E[p]$.*

This last corollary brings to light the parallelism between the conditional evaluation $E(\cdot | \alpha)$ and the well-known conditional expectation in probability theory [7]. As it happens in probability theory, the conditional evaluation is an element of $\mathbb{R}[x][\mathcal{A}]$ and, if we take the overall evaluation of $E(p | \alpha)$, this gives $E[p]$.

The conditional evaluation with respect to the auxiliary umbra $n \cdot \alpha$ is such that $E[(n + 1) \cdot \alpha | n \cdot \alpha] = E(n \cdot \alpha + \alpha' | n \cdot \alpha) = n \cdot \alpha + E[\alpha']$, with α' an umbra similar to α . By similar arguments, for all nonnegative integers n and m we have

$$E([(n + m) \cdot \alpha]^k | n \cdot \alpha) = E[(n \cdot \alpha + m \cdot \alpha']^k | n \cdot \alpha) = \sum_{j=0}^k \binom{k}{j} (n \cdot \alpha)^j E[(m \cdot \alpha')^{k-j}],$$

and by taking the evaluation of both sides we recover $E([(n + m) \cdot \alpha]^k)$ due to distributive property (2.2). Therefore, for $t \geq 0$ we define the conditional evaluation of $t \cdot \alpha$ with respect to the auxiliary umbra $s \cdot \alpha$, with $0 \leq s \leq t$ such as

$$E[(t \cdot \alpha)^k | s \cdot \alpha] = \sum_{j=0}^k \binom{k}{j} (s \cdot \alpha)^j E[(t - s) \cdot \alpha']^{k-j}. \quad (3.1)$$

Definition 3.4. *Let $\{P(x, t)\} \in \mathbb{R}[x]$ be a family of polynomials indexed by $t \geq 0$. $P(x, t)$ is said to be a time-space harmonic polynomial with respect to the family of auxiliary umbrae $\{q(t)\}_{t \geq 0}$ if and only if $E[P(q(t), t) | q(s)] = P(q(s), s)$ for all $0 \leq s \leq t$.*

Theorem 3.5. For all nonnegative integers k , the family of polynomials

$$Q_k(x, t) = E[(x - t.\alpha)^k] \in \mathbb{R}[x] \quad (3.2)$$

is time-space harmonic³ with respect to $\{t.\alpha\}_{t \geq 0}$.

Proof. From (3.2), by applying the linearity property of the evaluation E , we have

$$Q_k(x, t) = \sum_{j=0}^k \binom{k}{j} x^{k-j} E[(-t.\alpha)^j] \quad (3.3)$$

for all nonnegative integers k . Thanks to (3.1) and (2.2), we have

$$\begin{aligned} E(Q_k(t.\alpha, t) \mid s.\alpha) &= \sum_{j=0}^k \binom{k}{j} E[(t.\alpha)^{k-j} \mid s.\alpha] E[(-t.\alpha)^j] \\ &= \sum_{j=0}^k \binom{k}{j} \left\{ \sum_{i=0}^{k-j} \binom{k-j}{i} (s.\alpha)^i E(\{(t-s).\alpha'\}^{k-j-i}) \right\} E[(-t.\alpha)^j]. \end{aligned}$$

By suitably rearranging the terms, we have

$$\begin{aligned} E(Q_k(t.\alpha, t) \mid s.\alpha) &= \sum_{j=0}^k \binom{k}{j} (s.\alpha)^j \left\{ \sum_{i=0}^{k-j} \binom{k-j}{i} E(\{(t-s).\alpha'\}^{k-j-i}) E[(-t.\alpha)^i] \right\} \\ &= \sum_{j=0}^k \binom{k}{j} (s.\alpha)^j E(\{-t.\alpha + (t-s).\alpha'\}^{k-j}) = \sum_{j=0}^k \binom{k}{j} (s.\alpha)^j E[(-s.\alpha)^{k-j}]. \end{aligned}$$

□

Corollary 3.6. If $Q_k(x, t) = \sum_{j=0}^k q_j^{(k)}(t) x^j$, then

i) $q_j^{(k)}(t) = \binom{k}{j} E[(-t.\alpha)^{k-j}]$, for $t > 0$ and $j = 0, 1, \dots, k$; in particular $q_k^{(k)}(t) = 1$;

ii) $q_j^{(k)}(0) = 0$ for $j = 0, 1, \dots, k-1$ and $q_k^{(k)}(0) = 1$.

In particular we have $Q_k(x, 0) = x^k$ for all nonnegative integers k .

Proof. Property i) follows from (3.3). Property ii) follows by observing that $q_j^{(k)}(0) = \binom{k}{j} E[(0.\alpha)^{k-j}] = E[\epsilon^{k-j}]$ for $j = 0, 1, \dots, k-1$ and $q_k^{(k)}(0) = E(\epsilon^0) = 1$. □

The sequence of polynomials $\{Q_k(x, t)\}$ is umbrally represented by the polynomial umbra $x - t.\alpha$. We call $x - t.\alpha$ the *time-space harmonic polynomial umbra* with respect to $t.\alpha$.

Proposition 3.7. The time-space harmonic polynomial umbra $x - t.\alpha$ is the Appell umbra of $-t.\alpha$.

³When no confusion occurs, we will use the notation $x - t.\alpha$ to denote the polynomial umbra $-t.\alpha + x = x + (-t.\alpha)$.

The result follows by the definition of Appell umbra given in [4]. In particular Proposition 3.7 means that the sequence of polynomials $\{Q_k(x, t)\}$ is an Appell sequence, that is

$$\frac{d}{dx}Q_k(x, t) = kQ_{k-1}(x, t), \quad \text{for all integers } k \geq 1.$$

Let us observe that every linear combination of polynomials $\{Q_k(x, t)\}_{k \geq 1}$ is a time-space harmonic polynomial with respect to $\{t.\alpha\}_{t \geq 0}$.

Remark 3.8. Note that the family of auxiliary umbrae $\{t.\alpha\}_{t \in I}$, with $I \subset \mathbb{R}^+$, is the umbral counterpart of a stochastic process $\{X_t\}_{t \in I}$ such that $E[X_t^k] = E[(t.\alpha)^k]$ for all nonnegative integers k . This stochastic process is a Lévy process, see next section and in particular Remark 4.5 for a parallelism between the definition of $t.\alpha$ and the infinite divisible property of a Lévy process. Therefore, the polynomials $\{Q_k(x, t)\}$ are time-space harmonic with respect to Lévy processes. If the moments of $\{X_t\}_{t \in I}$ are defined only up to some finite m , the representation (3.2) still holds up to m , because it involves moments of order less or equal to m , see also next remark.

Remark 3.9. The *generating function* of an umbra α is the formal power series $f(\alpha, z) = \sum_{n \geq 0} a_n \frac{z^n}{n!} \in \mathbb{R}[x][[z]]$ whose coefficients are the moments of the umbra, see [2] for more details. Formal power series allow us to work with generating functions which do not have a positive radius of convergence or having undefined coefficients [18]. The generating function of the time-space harmonic polynomial umbra $x - t.\alpha$ is

$$f(x - t.\alpha, z) = \frac{\exp\{xz\}}{f(\alpha, z)^t} = \sum_{k \geq 0} Q_k(x, t) \frac{z^k}{k!}. \quad (3.4)$$

By replacing x with $t.\alpha$ in (3.4), we recover the Wald's exponential martingale (1.1). Equality of two formal power series is interpreted as the equality of their coefficients, so that $E[R_k(X_t, t)] = E[Q_k(t.\alpha, t)]$.

Also the Wald's identity $\sum_{k \geq 0} E[R_k(X_t, t)] z^k / k! = 1$ follows from (1.1). Therefore, the sequence $\{E[R_k(X_t, t)]\}_{k \geq 0} \in \mathbb{R}$ is umbrally represented by the augmentation umbra ϵ , with $f(\epsilon, z) = 1$, for all $t \geq 0$. But this is exactly what it happens when in the polynomial umbra $x - t.\alpha$ we replace x with $t.\alpha$.

The following corollary specifies the dependence of the coefficients of $Q_k(x, t)$ in (3.2) on the umbra α .

Corollary 3.10. *If $\{a_n\}$ is the sequence umbrally represented by the umbra α and $\{Q_k(x, t)\}$ are time-space harmonic polynomials with respect to $\{t.\alpha\}_{t \geq 0}$, then $Q_k(x, t) = \sum_{j, i=0}^k c_{i,j}^{(k)} t^i x^j$, with*

$$c_{i,j}^{(k)} = \binom{k}{j} \sum_{\lambda \vdash k-j} d_\lambda (-1)^{2l(\lambda)+i} s[l(\lambda), i] a_1^{r_1} a_2^{r_2} \dots$$

where $s[l(\lambda), i]$ are the Stirling numbers of first kind, the sum is over all partitions⁴ $\lambda = (1^{r_1}, 2^{r_2}, \dots)$ of the integer $k - j$ and $d_\lambda = i! / (r_1! r_2! \dots (1!)^{r_1} (2!)^{r_2} \dots)$.

⁴Recall that a partition of an integer i is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$, where λ_j are weakly decreasing positive integers such that $\sum_{j=1}^m \lambda_j = i$. The integers λ_j are named *parts* of λ . The *length* of λ is the number of its parts and will be indicated by $l(\lambda)$. A different notation is $\lambda = (1^{r_1}, 2^{r_2}, \dots)$, where r_j is the number of parts of λ equal to j and $r_1 + r_2 + \dots = l(\lambda)$. Note that r_j is said to be the multiplicity of j . We use the classical notation $\lambda \vdash i$ to denote “ λ is a partition of i ”.

Proof. Equation (2.1), with n replaced by t , can be restated as $E[(-t.\alpha)^i] = \sum_{\lambda \vdash i} d_\lambda(t) l(\lambda) a_1^{r_1} a_2^{r_2} \cdots$, see [3]. In particular we have $E[(-t.\alpha)^{k-j}] = \sum_{\lambda \vdash k-j} d_\lambda \left(\sum_{i=0}^{l(\lambda)} s[l(\lambda), i] (-1)^{i+2l(\lambda)} t^i \right) a_1^{r_1} a_2^{r_2} \cdots$, where $s[l(\lambda), i]$ is the i -th Stirling number of the first kind. From Corollary 3.6 we have

$$Q_k(x, t) = \sum_{j=0}^k \binom{k}{j} \left(\sum_{\lambda \vdash k-j} d_\lambda a_1^{r_1} a_2^{r_2} \cdots \sum_{i=0}^{l(\lambda)} (-1)^{i+2l(\lambda)} s[l(\lambda), i] t^i \right) x^j \quad (3.5)$$

and the result follows by suitably rearranging the terms in (3.5) and by observing that $s[l(\lambda), i] = 0$ for $i > l(\lambda)$. \square

In the following, assume $Q_k(x, t)$ in (3.2) such that $Q_k(x, t) = \sum_{j=0}^k q_j^{(k)}(t) x^j$ and denote by $\{a_n\}$ the sequence of moments umbrally represented by the umbra α in (3.2).

Proposition 3.11. *We have $q_j^{(k)}(t-1) = \sum_{i=j}^k \binom{i}{j} q_i^{(k)}(t) a_{i-j}$.*

Proof. From Corollary 3.6, we have $q_j^{(k)}(t-1) \simeq \binom{k}{j} [-(t-1).\alpha]^{k-j} \simeq \binom{k}{j} (-t.\alpha + \alpha')^{k-j} \simeq \binom{k}{j} \sum_{s=0}^{k-j} \binom{k-j}{s} (-t.\alpha)^{k-j-s} \alpha^s \simeq \sum_{i=j}^k \binom{i}{j} \binom{k}{i} (-t.\alpha)^{k-i} \alpha^{i-j}$. The result follows by taking the evaluation E of both sides. \square

Proposition 3.12. *We have $a_k = q_0^{(k)}(t-1) - \sum_{j=0}^{k-1} q_j^{(k)}(t) a_j$.*

Proof. By using Proposition 3.11, we have $q_0^{(k)}(t-1) = a_k q_k^{(k)}(t) + \sum_{j=0}^{k-1} a_j q_j^{(k)}(t)$. We have $q_0^{(k)}(t-1) - \sum_{j=0}^{k-1} a_j q_j^{(k)}(t) = a_k q_k^{(k)}(t) + \sum_{j=0}^{k-1} a_j q_j^{(k)}(t) - \sum_{j=0}^{k-1} a_j q_j^{(k)}(t) = a_k q_k^{(k)}(t)$. The result follows by observing that $q_k^{(k)}(t) = 1$. \square

Theorem 3.13. *A polynomial $P(x, t) = \sum_{j=0}^k p_j(t) x^j$, of degree k for all $t \geq 0$, is a time-space harmonic polynomial with respect to $\{t.\alpha\}_{t \geq 0}$ if and only if*

$$p_j(t) = \sum_{i=j}^k \binom{i}{j} p_i(0) E[(-t.\alpha)^{i-j}], \quad \text{for } j = 0, \dots, k. \quad (3.6)$$

Proof. Assume $P(x, t) = \sum_{j=0}^k p_j(t) x^j$ a polynomial whose coefficients satisfy (3.6). Then we have

$$\sum_{j=0}^k \left\{ \sum_{i=j}^k \binom{i}{j} p_i(0) E[(-t.\alpha)^{i-j}] \right\} x^j = \sum_{j=0}^k p_j(0) \sum_{i=0}^j \binom{j}{i} x^i E[(-t.\alpha)^{j-i}] = \sum_{j=0}^k p_j(0) E[(x-t.\alpha)^j]. \quad (3.7)$$

As $P(x, t)$ is a linear combination of $\{Q_k(x, t)\}$, then $P(x, t)$ is a time-space harmonic polynomial with respect to $\{t.\alpha\}_{t \geq 0}$. Viceversa if $P(x, t) = \sum_{j=0}^k p_j(t) x^j$ is a time-space harmonic polynomial with respect to $\{t.\alpha\}_{t \geq 0}$, then $P(x, t) = \sum_{i=0}^k c_i E[(x-t.\alpha)^i]$, with $\{c_i\} \in \mathbb{R}$. Therefore, from (3.7) and for $j = 0, \dots, k$ we have $p_j(t) = \sum_{i=j}^k \binom{i}{j} c_i E[(-t.\alpha)^{i-j}]$. So (3.6) follows by observing that, when t is replaced by 0, we have $p_j(0) = \sum_{i=j}^k \binom{i}{j} c_i E[(-0.\alpha)^{i-j}] = \sum_{i=j}^k \binom{i}{j} c_i E[\epsilon^{i-j}] = c_j$. \square

Corollary 3.14. *If $P(x, t) = \sum_{j=0}^k p_j(t) x^j$ is a polynomial of degree k for all $t \geq 0$, then there exists an umbra α such that $P(x, t)$ is a time-space harmonic polynomial with respect to $\{t.\alpha\}_{t \geq 0}$.*

4 Applications and examples

4.1 Discrete Case

When the parameter t is replaced by a nonnegative integer n , the coefficients $q_j^{(k)}(n)$ of time-space harmonic polynomials satisfy further properties thanks to the umbral representation (3.2). To keep the length of the paper within bounds, we just show some of them.

Proposition 4.1. *We have $q_j^{(k)}(n) + \sum_{i=j+1}^k \binom{i}{j} \sum_{l=1}^n q_i^{(k)}(l) a_{i-j} = 0$.*

Proof. From Corollary 3.6 $\sum_{i=j+1}^k \binom{i}{j} \sum_{l=1}^n \alpha^{i-j} q_i^{(k)}(l) \simeq \sum_{l=1}^n \sum_{i=j+1}^k \binom{i}{j} \alpha^{i-j} \binom{k}{i} (-l.\alpha)^{k-i}$. Since $\sum_{i=j+1}^k \binom{i}{j} \alpha^{i-j} \binom{k}{i} (-l.\alpha)^{k-i} \simeq \binom{k}{j} [(-l.\alpha + \alpha')^{k-j} - (-l.\alpha)^{k-j}] \simeq q_j^{(k)}(l-1) - q_j^{(k)}(l)$, the result follows by observing that $\sum_{i=j+1}^k \binom{i}{j} \sum_{l=1}^n \alpha^{i-j} q_i^{(k)}(l) \simeq \sum_{l=1}^n (q_j^{(k)}(l-1) - q_j^{(k)}(l)) \simeq q_j^{(k)}(0) - q_j^{(k)}(n) \simeq -q_j^{(k)}(n)$, since $q_j^{(k)}(0) \simeq \binom{k}{j} (-0.\alpha)^{k-j} \simeq 0$, for all $k \neq j$. \square

Corollary 4.2. *We have $q_0^{(k)}(n) + \sum_{l=1}^n \sum_{j=1}^k a_j q_j^{(k)}(l) = 0$.*

Remark 4.3. Let us observe that the family of umbrae $\{n.\alpha\}_{n \geq 0}$ corresponds to a discrete martingale $\{X_n\}_{n \geq 0}$ with $X_0 = 0$ and independent and identically distributed difference sequence with zero mean. Recall that the difference sequence associated to $\{X_n\}_{n \geq 0}$ is a sequence of random variables $\{M_n\}_{n \geq 0}$ such that $M_0 = X_0 = 0$ and $M_n = X_n - X_{n-1}$, for all nonnegative integers n . The umbra $n.\alpha$ generalizes $X_n = M_1 + M_2 + \dots + M_n$. Suppose to remove the identical distribution hypothesis on $\{M_n\}_{n \geq 0}$: in umbral terms, the martingale $\{X_n\}_{n \geq 0}$ corresponds to the umbra $\alpha_1 + \alpha_2 + \dots + \alpha_n$, where the umbrae $\{\alpha_1, \dots, \alpha_n\}$ are not necessarily similar. The time-space harmonic polynomials $E[(x - n.\alpha)^k]$ need to be replaced by $E[(x - 1.(\alpha_1 + \alpha_2 + \dots + \alpha_n))^k]$. The properties stated up to now can be recovered by similar arguments.

4.2 Cumulants

Any umbra is a partition umbra (cf. [2]). This means that if $\{a_n\}_{n \geq 0}$ is a sequence umbrally represented by an umbra α , then there exists a sequence $\{h_n\}_{n \geq 1}$ umbrally represented by an umbra κ_α , such that $\alpha \equiv \beta.\kappa_\alpha$. In terms of generating functions we have $f(\alpha, z) = \exp[f(\kappa_\alpha, z) - 1]$, so that $\{h_n\}_{n \geq 1}$ is the sequence of formal cumulants of $\{a_n\}_{n \geq 0}$ ⁵. The umbra κ_α is called α -cumulant umbra and we also have $\kappa_\alpha \equiv \chi.\alpha$, where χ is the singleton umbra.

Proposition 4.4. *For the sequence of polynomials $\{Q_k(x, t)\}$ umbrally represented by the time-space harmonic polynomial umbra $x - t.\alpha$, we have*

$$Q_k(x, t) = Y_k(x + h_1, h_2, \dots, h_k), \quad (4.1)$$

with Y_k exponential complete Bell polynomials and $\{h_n\}$ the sequence of cumulants of $-t.\alpha$.

Proof. We have $E[(\beta.\gamma)^k] = Y_k(g_1, g_2, \dots, g_k)$ with $g_n = E[\gamma^n]$, for all nonnegative integers n [2]. Therefore we will prove (4.1), if we show that $x + (-t.\alpha) \equiv \beta.\gamma$ for some polynomial umbra γ . Choose as umbra γ the umbra $\kappa_{(x.u)} \dot{+} \kappa_{(-t.\alpha)}$ with $\kappa_{(x.u)}$ the cumulant umbra of $x.u$ and $\kappa_{(-t.\alpha)}$

⁵In the ring of formal power series, given a sequence $\{a_n\}_{n \geq 1}$, its sequence of formal cumulants $\{h_n\}_{n \geq 1}$ is such that $1 + \sum_{n \geq 1} a_n z^n / n! = \exp\left(\sum_{n \geq 1} h_n z^n / n!\right)$.

the cumulant umbra of $-t.\alpha$. Let us recall that the disjoint sum of two distinct umbrae $\delta_1 \dot{+} \delta_2$ is an auxiliary umbra whose n -th moment is $E[\delta_1^n] + E[\delta_2^n]$ for all integers $n \geq 1$. We have

$$E[(\kappa_{(x.u)} \dot{+} \kappa_{(-t.\alpha)})^n] = \begin{cases} x + h_1 & n = 1 \\ h_n & n > 1 \end{cases}$$

with $\{h_n\}$ the sequence of cumulants of $-t.\alpha$. The result follows since $x + t.(-1.\alpha) \equiv \beta.\kappa_{(x.u)} + \beta.\kappa_{(-t.\alpha)} \equiv \beta.(\kappa_{(x.u)} \dot{+} \kappa_{(-t.\alpha)})$, [4]. \square

Remark 4.5. As $f(\beta.\kappa_\alpha, z) = \exp[f(\kappa_\alpha, z) - 1]$, the auxiliary umbra $\beta.\kappa_\alpha$ is the umbral counterpart of a compound Poisson random variable with parameter 1. More in general, compound Poisson random variables of parameter t are represented by the auxiliary umbrae $t.\beta.\kappa_\alpha$, with generating function $f(t.\beta.\kappa_\alpha, z) = \exp[t(f(\kappa_\alpha, z) - 1)]$ and we have $t.\alpha \equiv t.\beta.\kappa_\alpha$.

Let $\{X_t\}_{t \geq 0}$ be a real-value Lévy process, i.e. a process starting from 0 and with stationary and independent increments. If we denote the moment generating function of $X_{t+s} - X_s$ by $\phi(z, t)$, then $\phi(z, t)$ is infinitely divisible [12] and $\phi(z, t) = (\phi(z, 1))^t = (\psi(z))^t$, where $\psi(z)$ is the moment generating function of X_1 . In particular, $\phi(z, t) = \exp[t \log \psi(z)] = \exp[tk(z)]$, where $k(z)$ is the cumulant generating function of X_1 such that $k(0) = 0$. Comparing $\exp[tk(z)]$ with $\exp[t(f(\kappa_\alpha, z) - 1)]$, the correspondence between $t.\alpha \equiv t.\beta.\kappa_\alpha$ and the Lévy process $\{X_t\}_{t \geq 0}$ is immediate. We call $t.\alpha$ the *Lévy umbra* associated to the umbra α .

The following theorem explicitly states the connection between time-space harmonic polynomials with respect to Lévy processes $\{X_t\}_{t \geq 0}$ and the sequence of cumulants of X_1 .

Theorem 4.6. *For all nonnegative integers k , the family of polynomials*

$$Q_k(x, t) = E[(x - t.\beta.\kappa_\alpha)^k] \in \mathbb{R}[x] \quad (4.2)$$

is time-space harmonic with respect to $\{t.\alpha\}_{t \geq 0}$, where κ_α is the α -cumulant umbra.

The umbra $-t.\alpha \equiv t.(-1.\alpha)$ is the Lévy umbra associated to the umbra $-1.\alpha$. Therefore, also the polynomials $Q_k(x, t) = E[(x + t.\beta.\kappa_\alpha)^k]$ are time-space harmonic with respect to Lévy processes umbrally represented by $\{-t.\alpha\}_{t \geq 0}$. For simplicity, the following results refer to this last class of polynomials, but they can be stated also for the polynomials given in (4.2).

Moments of a polynomial umbra as $x + t.\beta.\gamma$ are generalizations of exponential complete Bell polynomials, see [4].

Proposition 4.7 (Sheffer identity with respect to t). *For the sequence of polynomials $\{Q_k(x, t)\}$ in (4.2) the following identity holds*

$$Q_k(x, t + s) = \sum_{j=0}^k \binom{k}{j} P_j(s) Q_{k-j}(x, t),$$

where $P_j(s) = Q_j(0, s)$ for all nonnegative integers j .

Proof. The result follows as $Q_k(x, t + s) = E[(x + t.\beta.\kappa_\alpha + s.\beta.\kappa_\alpha)^k] = \sum_{j=0}^k \binom{k}{j} E[(s.\beta.\kappa_\alpha)^j] E[(x + t.\beta.\kappa_\alpha)^{k-j}]$. \square

Corollary 4.8. *For the coefficients $\{q_j^{(k)}(t)\}$ of the sequence of polynomials $\{Q_k(x, t)\}$ in (4.2), we have*

$$\frac{d}{dt}q_j^{(k)}(t) = \sum_{i=1}^{k-j} \binom{k}{i} h_i q_j^{(k-i)}(t)$$

for $j = 1, \dots, k$, where $\{h_i\}$ is the sequence of cumulants of α .

Proof. Consider t as an indeterminate and observe that $\frac{d}{dt}q_j^{(k)}(t) = \binom{k}{j} \frac{d}{dt} E[(t.\beta.\kappa_\alpha)^{k-j}]$. We have

$$\frac{d}{dt}q_j^{(k)}(t) \simeq \binom{k}{j} \left\{ [(t + \chi).\beta.\kappa_\alpha]^{k-j} - (t.\beta.\kappa_\alpha)^{k-j} \right\} \simeq \binom{k}{j} \sum_{i=1}^{k-j} \binom{k-j}{i} \kappa_\alpha^i (t.\beta.\kappa_\alpha)^{k-j-i}, \quad (4.3)$$

see [4] for the first equivalence. The result follows by taking the evaluation of both sides in (4.3). \square

4.3 Special families of polynomials

In this section we show that some classical families of time-space harmonic polynomials are moments of the polynomial umbra $x - t.\alpha$, or can be recovered as a linear combination of its moments, for a suitable umbra α .

Hermite polynomials. Standard Brownian motion is a special Lévy process. From the Lévy-Khintchine formula [12], its generating function is $\phi(z, t) = \exp(tz^2/2)$. Therefore, the umbral counterpart of a standard Brownian motion is the umbra $t.\beta.\delta$, where $f(\delta, z) = 1 + z^2/2$. From Theorem 4.6, time-space harmonic polynomials with respect to a standard Brownian motion are $Q_k(x, t) = E[(x - t.\beta.\delta)^k]$.

Proposition 4.9. *For all nonnegative integers k we have $Q_k(x, t) = H_k^{(t)}(x)$, where $H_k^{(s^2)}(x)$ are the generalized Hermite polynomials with generating function $\sum_{k \geq 0} H_k^{(s^2)}(x) \frac{z^k}{k!} = \exp\{xz - s^2 \frac{z^2}{2}\}$.*

Proof. We have $H_k^{(s^2)}(x) = E[(x - 1.\beta.(s\delta))^k]$, see [5]. The result follows by observing that $-1.\beta.(\sqrt{t}\delta) \equiv -t.\beta.\delta$. \square

Poisson-Charlier polynomials. It is known that the Poisson-Charlier polynomials $\tilde{C}_k(x + t, t)$ are time-space harmonic with respect to the compensated Poisson process [17], with $\{\tilde{C}_k(x, t)\}$ polynomials having generating function $\sum_{k \geq 0} \tilde{C}_k(x, t) \frac{z^k}{k!} = e^{-tz}(1+z)^x$. We will recover this result, by proving that $\tilde{C}_k(x + t, t)$ is a linear combination of the polynomials $Q_k(x, t)$ in (3.2) when they are time-space harmonic with respect to the umbral counterpart of a compensated Poisson process.

Let $\{N_t\}_{t \geq 0}$ be a Poisson process of intensity 1 and $X_t = N_t - t$ the compensated process [13]. From the Lévy-Khintchine formula [12], its generating function is $\phi(z, t) = \exp[t(e^z - z)]$. The umbra γ having generating function $f(\gamma, z) = e^z - z$ is the disjoint difference of the unity umbra u and the singleton umbra χ . Indeed recall that given two distinct umbrae δ_1 and δ_2 , their *disjoint difference* is an auxiliary umbra, denoted by the symbol $\delta_1 \dot{-} \delta_2$, with generating function $f(\delta_1, z) - f(\delta_2, z) + 1$. Therefore, the umbral counterpart of a compensated Poisson process is $t.\beta.(u \dot{-} \chi)$. From Theorem 4.6, time-space harmonic polynomials with respect to a compensated Poisson process are $Q_k(x, t) = E[\{x - t.\beta.(u \dot{-} \chi)\}^k]$.

Proposition 4.10. *We have $\tilde{C}_k(x+t, t) = \sum_{j=1}^k s(k, j)Q_j(x, t)$, where $s(k, j)$ are the Stirling numbers of the first kind.*

Proof. By comparing generating functions, the sequence of polynomials $\{\tilde{C}_k(x, t)\}_{k \geq 0}$ is umbrally represented by the polynomial umbra $x.\chi - t$. Therefore, we have $\tilde{C}_k(x+t, t) = E[(x+t).\chi - t]^k$. We also have $(x+t).\chi - t \equiv (x.\chi - t) + t.\chi \equiv ((x.\chi - t).\beta + t).\chi$, due to the distributive property (2.4) and $\beta.\chi \equiv u$. Since $E[(\alpha.\chi)^k] = E[(\alpha)_k]$, see [2], we have $\tilde{C}_k(x+t, t) = E[((x.\chi - t).\beta + t)_k] = \sum_{j=0}^k s(k, j)E[(x.\chi - t).\beta + t]^j$. The result follows by showing that $x - t.\beta.(u - \chi) \equiv (x.\chi - t).\beta + t$. Indeed we have $x - t.\beta.(u - \chi) \equiv x - t.\beta - t.\beta.(-\chi) \equiv (x - t.\beta - t.\beta.(-\chi)).\chi.\beta \equiv (x.\chi - t - t.\beta.(-\chi).\chi).\beta$, where we have used again $\chi.\beta \equiv u$ and the distributive property (2.4). The result follows since $-\chi \equiv \chi.(-1)$ and $-t.\beta.(-\chi).\chi \equiv -t.\beta.\chi.(-1).\chi \equiv t.\chi$. \square

Lévy-Sheffer polynomials. According to the definition given in [14], a sequence of polynomials $\{V_k(x, t)\}_{t \geq 0}$ is a Lévy-Sheffer system if it is defined by the following generating function

$$\sum_{k \geq 0} V_k(x, t) \frac{z^k}{k!} = (g(z))^t \exp\{xu(z)\}, \quad (4.4)$$

where $g(z)$ and $u(z)$ are analytic in a neighborhood of $z = 0$, $u(0) = 0$, $g(0) = 1$, $u'(0) \neq 0$ and $1/g(\tau(z))$ is an infinitely divisible moment generating function, with $\tau(z)$ such that $\tau(u(z)) = z$. Schoutens in [14] states that the basic link between these polynomials and Lévy processes is a *martingale equality* (cf. pag. 337 eq. (6)), which is equivalent to ask that these polynomials are time-space harmonic with respect to Lévy processes.

In the following, we show that $V_k(x, t)$ is a linear combination of suitable time-space harmonic polynomials $Q_k(x, t)$ and therefore, they share the same property.

To this aim we need to recall the notion of compositional inverse of an umbra α . Recall first that the auxiliary umbra $\alpha.\beta.\gamma$ is the *composition umbra* of α and γ , see [2]. The name recalls that its generating function is the composition of $f(\alpha, z)$ and $f(\gamma, z)$, that is $f(\alpha.\beta.\gamma, z) = f(\alpha, f(\gamma, z) - 1)$. Moreover its moments are

$$E[(\alpha.\beta.\gamma)^i] = \sum_{k=1}^i a_k B_{i,k}(g_1, \dots, g_{i-k+1}). \quad (4.5)$$

If $E[\alpha] \neq 0$, the compositional inverse of an umbra α is the auxiliary umbra $\alpha^{<-1>}$ such that $\alpha.\beta.\alpha^{<-1>} \equiv \alpha^{<-1>}.\beta.\alpha \equiv \chi$, with χ the singleton umbra.

Theorem 4.11. *We have $V_k(x, t) = \sum_{i=0}^k E[(x + t.\beta.\kappa)^i] B_{k,i}(g_1, \dots, g_{k-i+1})$, where $g_j = E[\gamma^j]$, for all nonnegative j and κ is the cumulant umbra of $\alpha.\beta.\gamma^{<-1>}$, with $\gamma^{<-1>}$ the compositional inverse of the umbra γ .*

Proof. Due to the form of the generating function in (4.4), the polynomials $\{V_k(x, t)\}_{k \geq 0}$ are umbrally represented by the polynomial umbra $t.\alpha + x.\beta.\gamma$, with $f(\alpha, z) = g(z)$ and $f(\gamma, z) = 1 + u(z)$, and $E[\gamma] = u'(0) \neq 0$. Thanks to the distributive property (2.4) we have $x.\beta.\gamma + t.\alpha \equiv (x + t.\alpha.\beta.\gamma^{<-1>}).\beta.\gamma$. The result follows by using (4.5). \square

Corollary 4.12. *The Lévy-Sheffer polynomials $\{V_k(x, t)\}_{t \geq 0}$ are time-space harmonic with respect to Lévy processes umbrally represented by $\{-t.\alpha.\beta.\gamma^{<-1>}\}_{t \geq 0}$.*

Time-space harmonic polynomials in terms of boolean and free cumulants. Let $M(z)$ be the ordinary generating function of a random variable X , that is $M(z) = 1 + \sum_{i \geq 1} a_i z^i$, where $a_i = E[X^i]$. We have $M(z) = 1/[1 - B(z)]$, where $B(z) = \sum_{i \geq 1} b_i z^i$, and b_i are the boolean cumulants of X . The noncrossing (or free) cumulants of X are the coefficients r_i of the ordinary power series $R(z) = 1 + \sum_{i \geq 1} r_i z^i$ such that $M(z) = R[zM(z)]$. The umbral theory of boolean and free cumulants has been introduced in [6]. In particular, the α -boolean cumulant umbra η_α has moments $E[\eta_\alpha^i] = b_i$ for all nonnegative integers i . This umbra is such that $\bar{\alpha} \equiv \bar{u}.\beta.\bar{\eta}_\alpha$, where $E[\bar{u}^n] = n!$ and $E[\bar{\alpha}^n] = n!a_n$ for all nonnegative integers n . Thanks to Theorem 3.5, the polynomials $Q_k(x, t) = E[(x - t.\bar{u}.\beta.\bar{\eta}_\alpha)^k]$ are time-space harmonic polynomials with respect to the family $\{t.\bar{\alpha}\}_{t \geq 0}$ with η_α the α -boolean cumulant umbra. The $\bar{\alpha}$ -free cumulant $\bar{\mathfrak{R}}_{\bar{\alpha}}$ has moments $E[\bar{\mathfrak{R}}_{\bar{\alpha}}^i] = i!r_i$, for all nonnegative integers i . This umbra allows us a different parametrization of $Q_k(x, t) = E[(x - t.\bar{\alpha})^k]$. Indeed, if we denote by $\bar{\alpha}_D$ the derivative umbra of $\bar{\alpha}$, such that $f(\bar{\alpha}_D, z) = 1 + zf(\bar{\alpha}, z)$, then we have $\bar{\alpha} \equiv \bar{\mathfrak{R}}_{\bar{\alpha}}.\beta.(-1.\bar{\mathfrak{R}}_{\bar{\alpha}})^{<-1>}_D$. Therefore, also the polynomials $Q_k(x, t) = E[(x + t.(-1.\bar{\mathfrak{R}}_{\bar{\alpha}}).\beta.(-1.\bar{\mathfrak{R}}_{\bar{\alpha}})^{<-1>}_D)^k]$ are time-space harmonic polynomials with respect to the family $\{t.\bar{\alpha}\}_{t \geq 0}$.

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